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SOLUTIONS, WITH A DEGENERATE HODOGRAPH, OF QUASISTEADY EQUATIONS OF THE THEORY OF PLASTICITY WITH THE VON MISES YIELD CONDITION

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Simple waves are often used from solutions with a degenerate hodograph in the theory of plasticity when the system of equations which describes plastic flow is hyperbolic and has two independent variables. There are only isolated instances of the construction of such solutions in a plastic body when the number of independent variables is greater than two. In this study, we present a complete classification of double waves in the case of a plastic–rigid body described by quasisteady equations characterized by functional arbitrariness

$$\partial\sigma/\partial x_i + \partial S_{i\alpha}/\partial x_\alpha = 0; \quad (1)$$

$$\operatorname{div} \mathbf{v} = 0; \quad (2)$$

$$\partial v_i/\partial x_j + \partial v_j/\partial x_i = 2\Psi S_{ij}\ (i,\ j = 1,\ 2,\ 3) \quad (3)$$

with the von Mises yield criterion

$$S_{\alpha\beta}S_{\alpha\beta} = 2k^2. \quad (4)$$

Here, (S_{ij}) is the deviator of the stress tensor $(S_{\alpha\alpha} = 0)$; $v = (v_1, v_2, v_3)'$ is the vector of the rate of displacement; σ is the normal stress; k is the yield point in shear; Ψ is the proportionality factor in the associated flow law; summation is performed from 1 to 3 over the repeating Greek-letter indices. Without loss of generality, we take $S_1 \neq 0$ ($S_i \equiv S_{ii}$, $i = 1, 2, 3$, $S_3 = -S_1 - S_2$).

Equations (3) are inhomogeneous. Since $S_1 \neq 0$, from (3) at i = j = 1 we find $\Psi = \frac{1}{S_1}\frac{\partial v_1}{\partial x_1}$. After we exclude Ψ from the remaining equations of (3), we obtain a closed homogeneous system of nine quasilinear differential equations relative to nine unknowns: (1), (2), (4), and

$$S_1(\partial v_i/\partial x_j + \partial v_j/\partial x_i) - 2S_{ij}\partial v_1/\partial x_1 = 0\ \ (i,\ j = 1,\ 2,\ 3). \quad (5)$$

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For solutions of system (1)-(2), (4)-(5) of the double-wave type, characterized by functional arbitrariness in the general solution of the Cauchy problem, there are only two possibilities: either functions v_1, v_2 such that the Jacobian $\partial(v_1, v_2)/\partial(x_1, x_2) \neq 0$, or $v_i = v_i(v_1)$ $(i = 2, 3)$.

1. Let the Jacobian $\partial(v_1, v_2)/\partial(x_1, x_2) \neq 0$. In this case, the functions v_1, v_2 are chosen as the parameters of the double wave and all of the remaining parameters (σ, $S_{i,j}$, v_3) are chosen through them:

$$v_3 = v_3(v_1, v_2),\ \sigma = \sigma(v_1, v_2),\ S_{ij} = S_{ij}(v_1, v_2) \quad (i, j = 1, 2, 3). \tag{6}$$

Insertion of these expressions into (1)-(5) gives us a homogeneous system of quasilinear first-order differential equations $G\mathbf{f} = 0$ with the matrix G, column vector $\mathbf{f} = (v_{1,3}, v_{2,3}, v_{1,1}, v_{2,1}, v_{1,2}, v_{2,2})'$, and notation $v_{i,j} = \partial v_i/\partial x_j$, $v_{3,i} = \partial v_3/\partial v_i$, $\sigma_i = \partial\sigma/\partial v_i$, $S_{kj,i} = \partial S_{kj}/\partial v_i$, $v_{3,in} = \partial^2 v_3/(\partial v_i \partial v_n)$ $(i, n = 1, 2;\ k, j = 1, 2, 3)$.

To ensure that there is no reduction to invariant solutions [1], it is necessary to require satisfaction of the inequality rank $G \leq 4$. Since $S_1 \neq 0$, it follows from the form of the matrix G that rank $G \geq 4$. Thus, for double waves that are not reducible to invariant solutions, the condition rank $G = 4$ is satisfied.

If we use a_{ij} to represent the determinant of a fifth-order matrix composed of elements of the matrix G standing at the intersection of the first four rows and the i-th row with the first four columns and the j-th column, then

$$a_{ij} = 0,\ 5 \leqslant i \leqslant 8, \tag{7}$$

while the four independent equations of the system $G\mathbf{f} = 0$ have the form

$$\partial\mathbf{u}/\partial x_2 = G_2\partial\mathbf{u}/\partial x_1; \tag{8}$$

$$\partial\mathbf{u}/\partial x_3 = G_1\partial\mathbf{u}/\partial x_1, \tag{9}$$

where

$$\mathbf{u} = (v_1, v_2)';\ \xi \equiv (2S_{23} - 2S_{12}v_{3,1} - S_2 v_{3,2})/S_1;$$
$$G_2 = \begin{pmatrix} 2S_{12}/S_1 & -1 \\ S_2/S_1 & 0 \end{pmatrix},\quad G_1 = \begin{pmatrix} 2S_{13}/S_1 - v_{3,1} & -1 \\ \xi & 0 \end{pmatrix}.$$

Certain equations $a_{ij} = 0$ give the relations for finding the function $\sigma = \sigma(v_1, v_2)$:

$$\begin{aligned} \sigma_1 &= -S_{23,1}v_{3,2} + S_{23,2}v_{3,1} + S_{12,2} - S_{2,1}, \\ \sigma_2 &= S_{13,1}v_{3,2} - S_{13,2}v_{3,1} + S_{12,1} - S_{1,2}. \end{aligned} \tag{10}$$

After exclusion of the second derivatives $\partial^2\mathbf{u}/\partial x_i \partial x_j$ (i, j = 1, 2, 3, in addition to i = j = 1), $(G_1G_2 - G_2G_1)\,\partial^2\mathbf{u}/\partial x_1^2 = \Phi(\mathbf{u}, \partial\mathbf{u}/\partial x_1)$ remains in extended system (8)-(9). This means that the maximum possible arbitrariness in the solution of system (8)-(9) with assigned functions (6) is determined by the number $2 - r$ ($r \equiv$ rank $(G_1G_2 - G_2G_1)$). Thus, it is necessary that $r \leq 1$ in order for an ideal plastic–rigid body to contain double waves having functional arbitrariness in the Cauchy solution and being perpendicular to the invariant solutions.

We will henceforth write the expression for the matrix

$$G_1G_2 - G_2G_1 = \frac{2}{S_1}\begin{pmatrix} Z_1 & Z_2 \\ (S_2Z_2 + S_{12}Z_1)/S_1 & -Z_1 \end{pmatrix},$$

where $Z_1 = S_{23} - S_{12}v_{3,1} - S_2v_{3,2}$; $Z_2 = -S_{13} + S_1v_{3,1} + S_{12}v_{3,2}$.

Let us examine the cases $r = 0$ and $r = 1$ in succession.

1.1. Let $r = 0$. Then $G_1G_2 - G_2G_1 = 0$ and, thus, $Z_1 = 0$, $Z_2 = 0$ or

$$S_{23} = S_{12}v_{3,1} + S_2v_{3,2},\ S_{13} = S_1v_{3,1} + S_{12}v_{3,2}. \tag{11}$$

If we completely differentiate D_3 of Eq. (8) with respect to x_3, subtract from the result Eq. (9) after D_2 has been completely differentiated with respect to x_2, and substitute the derivatives $\partial \mathbf{u}/\partial x_2$, $\partial \mathbf{u}/\partial x_3$ from (8) and (9), then with allowance for (11) we obtain two homogeneous invariant forms relative to the first derivatives $\partial \mathbf{u}/\partial x_1$:

$$g(S_{12}v_{3,22} + S_1 v_{3,12}) = 0,\ g(S_1 v_{3,11} - S_2 v_{3,22}) = 0. \tag{12}$$

Here, $g \equiv S_2\left(\frac{\partial v_1}{\partial x_1}\right)^2 - 2S_{12}\frac{\partial v_1}{\partial x_1}\frac{\partial v_2}{\partial x_1} + S_1\left(\frac{\partial v_2}{\partial x_1}\right)^2$.

It follows from the prohibition on the reduction to an invariant solution that $g \neq 0$. However, we then find the following from (12)

$$v_{3,12} = -S_{12}v_{3,22}/S_1,\ v_{3,11} = S_2 v_{3,22}/S_1. \tag{13}$$

If $v_{3,22} = 0$, then $v_3 = c_1v_1 + c_2v_2 + c_3$ (c_i = const). By rotating coordinate axes and shifting, such a solution is reduced to plane deformation. Thus, $v_{3,22} \neq 0$.

Satisfaction of (13) ensures identical satisfaction of (12), the latter being necessary and sufficient conditions to ensure that redefined system (8)-(9) is involute. It should also be noted that the double wave in this case will be a solution with rectilinear generators.

It remains for us to analyze the compatability of Eqs. (4), (7), (11), and (13). Due to the cumbersome nature of these calculations, they were performed on a computer [5]. All of the components of the stress tensor are determined through the function $v_3(v_1, v_2)$ and its second-order derivatives; of Eqs. (7), only (10) will be independent, and all the remaining equations are satisfied identically ($a_{k\ell} \equiv 0$, $5 \leq k \leq 8$); we obtain a redefined system of two differential equations for the function $v_3(v_1, v_2)$: one second-order equation and one fourth-order equation. These equations are used to derive explicit expressions for all fifth derivatives of $v_3(v_1, v_2)$ with respect to the variables v_1 and v_2. Thus, $v_3(v_1, v_2)$ is found with a constant arbitrariness. We could not further analyze the compatibility of the given system, due to the volume of computation required and the limited computer memory available. The system is compatible, and one of its solutions, with arbitrariness in the form of two constants, was presented in [2]. The system apparently has no other solutions except those in [2].

1.2. Let r = 1. Then rank G = 1. This corresponds to satisfaction of the relations ($a \equiv hv_{3,2} - v_{3,1}$)

$$Z_1^2 + Z_2^2 \neq 0; \tag{14}$$

$$S_{23} + hS_{13} = -(S_1h + S_{12})a, \tag{15}$$

where

$$S_{12}^2 - S_1S_2 \geqslant 0;\ h = (-S_{12} \pm (S_{12}^2 - S_1S_2)^{1/2})/S_1.$$

Since $\partial(v_1, v_2)/\partial(x_1, x_2) \neq 0$, we make a transition to new independent variables (v_1, v_2, x_3):

$$x_1 = P(v_1, v_2, x_3),\ x_2 = Q(v_1, v_2, x_3). \tag{16}$$

Here, system (8)-(9) for the function $v_i(x_1, x_2, x_3)$ (i = 1, 2) changes into a system of differential equations for the function $P(v_1, v_2, x_3)$, $Q(v_1, v_2, x_3)$. After certain algebraic transformations, the new system reduces to the form

$$\begin{gathered} S_1P_1 - S_2Q_2 = 0,\ S_1P_2 + S_1Q_1 + 2S_{12}Q_2 = 0, \\ S_1(Q_3 + v_{3,2})Q_1 + (2S_{12}(Q_3 + v_{3,2}) + S_1(P_3 + v_{3,1}) - 2Z_2)Q_2 = 0, \\ S_1(P_3 + v_{3,1})Q_1 - (S_2(Q_3 + v_{3,2}) + 2Z_1)Q_2 = 0. \end{gathered} \tag{17}$$

Here $P_i = \partial P/\partial v_i$, $Q_i = \partial Q/\partial v_i$ (i = 1, 2); $P_3 = \partial P/\partial x_3$; $Q_3 = \partial Q/\partial x_3$;

$$P_1Q_2 - P_2Q_1 \neq 0. \tag{18}$$

System (17) is linear and homogeneous relative to P_i, Q_i ($i = 1, 2$). By virtue of inequality (18), its determinant must vanish, i.e., ($\gamma = \pm 1$)

$$S_1(P_3 + v_{3,1}) = Z_2 - S_{12}(Q_3 + v_{3,2}) + \gamma((S_1 h + S_{12})(Q_3 + v_{3,2}) - Z_2). \tag{19}$$

We then examine two variants $\gamma = -1$ and $\gamma = +1$.

A contradiction to condition (14) arises at $\gamma = -1$.

Let $\gamma = +1$. Then after integration of (19) with respect to x_3

$$P = hQ + x_3 a + \chi, \tag{20}$$

where $\chi = \chi(v_1, v_2)$ is an arbitrary function; $a_i \equiv \partial a/\partial v_i$; $\chi_i = \partial\chi/\partial v_i$; $h_i \equiv \partial h/\partial v_i$; $i = 1, 2$.

Insertion of (20) into (17) with allowance for (15) gives

$$Q(h_1 - hh_2) + x_3(a_1 - ha_2) + \chi_1 - h\chi_2 = 0; \tag{21}$$

$$S_1 Q_1 + (hS_1 + S_{12})Q_2 = -\psi; \tag{22}$$

$$Q_2(2S_{13} - S_1 v_{3,1} - v_{3,2}(hS_1 + S_{12})) - \psi Q_3 = \psi v_{3,2}; \tag{23}$$

$$(2Q_2(hS_1 + S_{12}) + \psi)a = 0 \ (\psi \equiv S_1(h_2 Q + x_3 a_2 + \chi_2)). \tag{24}$$

We will henceforth study two cases of system (21)-(24): $a \neq 0$ and $a = 0$.

If $a \neq 0$ and $hS_1 + S_{12} \neq 0$, then it follows from (22) and (24) that $Q_1 = hQ_2$. However, then a contradiction to condition (18) follows from (21) and (20). Thus, if $a \neq 0$, then $hS_1 + S_{12} = 0$. This means that $\psi = 0$. Thus, as in the preceding case, we obtain the contradiction (18).

As a consequence, $a = 0$. In accordance with (15),

$$S_{23} = -hS_{13}, \tag{25}$$

while from (7) and the definition of h we obtain

$$S_2 = -S_1; \tag{26}$$

$$S_{12} = S_1(1 - h^2)/(2h),\ h \neq 0. \tag{27}$$

Having inserted (25)-(27) into the von Mises condition (3), we write

$$S_1^2(1 + h^2)^2 + 4h^2(1 + h^2)S_{13}^2 = 4h^2k^2. \tag{28}$$

With allowance for (25)-(28), Eqs. (7) converge only to

$$h^2 S_{1,1} + hS_{1,2} + S_1 h_2(h^2 - 1) = 0. \tag{29}$$

Since $Q_3 \neq 0$, then we find from (21) that

$$h_1 = hh_2,\ \chi_1 = h\chi_2. \tag{30}$$

If $h_2 = 0$, then $h = \text{const}$. Analysis of the remaining two equations of (10) leads only to the case when $v_3 = c_1(v_2 + hv_1) + c_2$ (c_1, c_2 = const) (and this means a reduction to plane strain) or to the case when S_{ij} = const. Thus, it is necessary that $h_2 \neq 0$. Then from the condition $a = 0$ and from (30) we establish that $v_3 = v_3(h)$ and $\chi = \chi(h)$.

Since $h_2 \neq 0$, we can replace the variables (v_1, v_2) by (h, λ), where the function $\lambda(v_1, v_2)$ is such that

$$\lambda_1 = -1/(1 + h^2)^{1/2},\ \lambda_2 = h/(1 + h^2)^{1/2}. \tag{31}$$

In the new variables, Eqs. (22) and (29) take the form

$$h(1 + h^2)\partial S_1/\partial h + S_1(h^2 - 1) = 0; \tag{32}$$

$$(1 + h^2)\partial Q/\partial h + hQ = -h\chi'. \tag{33}$$

Integrating Eq. (32) over h, we find

$$S_1 = hc(\lambda)/(1 + h^2), \tag{34}$$

while the relations below follow from Eq. (27) and the von Mises criterion

$$S_{12} = (1 - h^2)c(\lambda)/(2(1 + h^2)),\ S_{13} = c_1(\lambda)/(1 + h^2)^{1/2}, \tag{35}$$

where $c(\lambda)$ is an arbitrary function and $c_1 = \pm(k^2 - c^2/4)^{1/2}$.

After we insert the resulting expressions for the components of the deviator of the stress tensor (S_{ij}) into (10), we have $\sigma = \sigma(h)$ and

$$\sigma' h_2 = -h_2 v_3' c_1' - ch_2/(1 + h^2) - c'/(2(1 + h^2)^{1/2}). \tag{36}$$

Equation (33) is also integrated:

$$Q = (g(h) + B(\lambda, x_3))/(1 + h^2)^{1/2}. \tag{37}$$

Here, $B(\lambda, x_3)$ is an arbitrary function; the function $g = g(h)$ is such that $g' = -h\chi'/(1 + h^2)^{1/2}$. Here, $P_1Q_2 - P_2Q_1 = h_2(Q + \chi')\partial B/\partial\lambda \neq 0$, while since $Q_3 \neq 0$, then $B + g + \chi'(1 + h^2)^{1/2} \neq 0$. As a result, (23) is changed to the form

$$\begin{gathered}\partial B/\partial\lambda\,(f(\lambda, h)/(B + g + \chi'(1 + h^2)^{1/2})) + c\,\partial B/\partial x_3 + 2c_1 = 0\\ (f \equiv cv_3'(1 + h^2) - 2c_1(1 + h^2)^{1/2}/h_2).\end{gathered} \tag{38}$$

After we differentiate (38) with respect to h and we use the condition $\partial B/\partial\lambda \neq 0$, we find $(\partial/\partial h)(f/(B + g + \chi'(1 + h^2)^{1/2})) = 0$. Since $\partial B/\partial x_3 \neq 0$, then $\partial f/\partial h = 0$ and $f(\partial/\partial h)(g + \chi'(1 + h^2)^{1/2}) = 0$.

The condition $f = 0$ leads to contradictory relations. Thus, we obtain the following from the last equations

$$\chi = c_3h + c_4,\ g = -c_3(1 + h^2)^{1/2}\ (c_3,\ c_4 = \text{const}). \tag{39}$$

The arbitrary constants c_3 and c_4 are immaterial. For example, $c_3 = c_4 = 0$.

To find $\partial h_2/\partial h$ in $\partial f/\partial h$, we use $\dfrac{\partial}{\partial h} = \dfrac{1}{h_2(1 + h^2)}\left(\dfrac{\partial}{\partial v_2} + h\dfrac{\partial}{\partial v_1}\right)$. Then

$$\partial f/\partial h = 2c_1h_{22}(1 + h^2)^{1/2}/h_2^3 + c\left(v_3''(1 + h^2) + 2hv_3'\right) = 0. \tag{40}$$

We will examine three cases: $c_1 = 0$; $c_1 \neq 0$, $c' = 0$; $c_1 \neq 0$, $c' \neq 0$.

A. It is assumed that $c_1 = 0$. Then we find from (36) that

$$\sigma' = -c/(1 + h^2),\quad v_3' = b/(1 + h^2),\quad c_1 = \pm 2k\ (b = \text{const}). \tag{41}$$

Here, $f = cb$, and it is easy to write the general solution of (38)

$$\lambda B = bx_3 + \Phi(B) \tag{42}$$

with the arbitrary function $\Phi(B)$. Thus, the general solution of the nonreducible double wave in the given case has two arbitrary functions with the same argument: one with the definition $h = h(v_1, v_2)$, the other with $\Phi = \Phi(B)$. Let us analyze this solution in the initial space of the independent variables (x_1, x_2, x_3).

The relation $h(v_1, v_2) = x_1/x_2$ follows from (16), (20), and (39). The use of (15), (37), (39), and (42) yields $\lambda(v_1, v_2) = \beta(bx_3 + \Phi(R))/R$, where $\beta = \mathrm{sgn}(x_2)$, $R = (x_1^2 + x_2^2)^{1/2}$. The expressions $h = h(v_1, v_2)$, $\lambda = \lambda(v_1, v_2)$ are used to implicitly find the components of velocity (v_1, v_2) as functions of the independent variables (x_1, x_2, x_3).

Having differentiated $h = h(v_1, v_2)$ and $\lambda = \lambda(v_1, v_2)$ with respect to x_3 and having made use of (30) and (31), we obtain $h\partial v_1/\partial x_3 + \partial v_2/\partial x_3 = 0$, $-\partial v_1/\partial x_3 + h\partial v_2/\partial x_3 = b/x_2$. It follows from this that

$$v_1 = -bx_2x_3/R^2 + g_1(x_1, x_2),\ v_2 = bx_1x_3/R^2 + g_2(x_1, x_2). \tag{43}$$

Before we find the function $g_i(x_1, x_2)$ $(i = 1, 2)$, we need to write the expressions for the stresses s_{ij}, σ, and v_3. To do this, by analogy with plane strain we introduce an angle θ such that $h = (\cos 2\theta - \gamma)/\sin 2\theta$ $(\gamma = \pm 1)$.

We obtain the following from (34), (35), and (41)

$$S_{13} = S_{23} = 0,\ S_1 = -S_2 = -k\sin 2\theta,\ S_{12} = k\cos 2\theta,\ v_3 = b\theta + c_s. \tag{44}$$

Substitution of (43)-(44) into (8)-(9) shows that (9) is satisfied identically, while (8) leads to

$$\frac{\partial g_1}{\partial x_2} + 2\,\mathrm{ctg}\,2\theta\,\frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_1} = 0,\quad \frac{\partial g_2}{\partial x_2} + \frac{\partial g_1}{\partial x_1} = 0.$$

The general solution of the last equations will be [3, 4]

$$g_1 = \left(\varphi_1 + \varphi_2 + h(1 + h^2)\varphi_1'\right)/(1 + h^2)^{1/2},$$
$$g_2 = \left((1 + h^2)\varphi_1' - h(\varphi_1 + \varphi_2)\right)/(1 + h^2)^{1/2}$$

while the arbitrary functions $\varphi_1 = \varphi_1(h)$, $\varphi_2 = \varphi_2(R)$.

<u>B.</u> Let $c_1 \neq 0$, $c' = 0$. Then from (36) $\sigma' = -c/(1 + h^2)$, while after integration of (40) over v_2

$$1/h_2 = (c/2c_1)\left(v_3'(1 + h^2)^{1/2} + \int \left(hv_3'/(1 + h^2)^{1/2}\right)dh + \mu(v_1)\right) \tag{45}$$

with the arbitrary function $\mu = \mu(v_1)$. The form of this function is obtained from study of the compatibility condition of the system of two differential equations for the function $h = h(v_1, v_2)$: the first equation of (30) and (45). This system turns out to be compatible only if $\mu = -2c_1v_1/c + c_5$ (c_5 is an arbitrary constant). Since the determination of the function $\lambda(v_1, v_2)$ is accurate to within the arbitrary constant, then $f = -2c_1\lambda$. Here, the general solution of Eq. (38) $\lambda = B\Phi(B + 2c_1x_3/c)$ with an arbitrary function $\Phi = \Phi(\zeta)$ having the argument $\zeta = B + 2c_1x_3/c$.

Thus, a nonreducible double wave is also associated with general randomness in case B – there are two functions with one argument: $v_3 = v_3(h)$ and $\Phi = \Phi(\zeta)$. The solution is constructed as follows in the initial space of independent variables (x_1, x_2, x_3): we first assign arbitrary functions $v_3(h)$ and $\Phi(\zeta)$; we then find the function $h = h(v_1, v_2)$ from the equation

$$\int \left(v_3'(1 + h^2)^{1/2} + \int \left(hv_3'/(1 + h^2)^{1/2}\right)dh + \mu(v_1)\right)dh - (2c_1/c)(hv_1 + v_2) = \text{const},$$

We then determine the function $\lambda(v_1, v_2) = -(c/2c_1)v_3'(1 + h^2) + (1 + h^2)^{1/2}/h_2$. Finally, we reconstruct the components of velocity $v_1(x_1, x_2, x_3)$, $v_2(x_1, x_2, x_3)$ from the equations $(\alpha = \mathrm{sgn}\,(x_2))$

$$h(v_1, v_2) = x_1/x_2,\ \lambda(v_1, v_2) = \alpha\left(x_1^2 + x_2^2\right)^{1/2}\Phi\left(\alpha\left(x_1^2 + x_2^2\right)^{1/2} + 2c_1x_3/c\right).$$

The stress state is represented by the relations (c_2 = const)

$$S_1 = -S_2 = hc/(1 + h^2),\quad S_{12} = (1 - h^2)c/(2(1 + h^2)),$$
$$S_{13} = c_1/(1 + h^2)^{1/2},\quad S_{23} = -hS_{13},\ \sigma = -c\,\mathrm{artg}\,(h) + c_2.$$

<u>C.</u> Let $c_1 \neq 0$, $c' \neq 0$. Then, having differentiated (36) with respect to $\partial/\partial h$ and having inserted it into the expression for h_{22} from (40), we obtain $\sigma'' + 2h\sigma'/(1 + h^2) = 0$. From this, $\sigma' = b/(1 + h^2)$ (b = const). After we substitute σ' into (36) and differentiate it with respect to $\partial/\partial\lambda$ (considering that $\partial/\partial\lambda = (h\partial/\partial v_2 - \partial/\partial v_1)/(1 + h^2)^{1/2}$ and $h_1 = hh_2$), we have $c''/h_2 + 3c'\ (1 + h^2)^{-1/2} + v_3'c_1''\ (1 + h^2)^{1/2} = 0$. Alternatively, we exclude h_2 from the last equation by means of (36), then we can write $c''(b + c) - (3/2)(c')^2 + k^2(c'/c_1')^3(v_3'(1 + h^2)) = 0$. Since $c(\lambda)$ and $c_1(\lambda)$, then $v_3'(1 + h^2) = b_1$ = const. We then find

from (40) that $h_{22} = 0$. This leads us to the relations $h = (c_4 - v_2)/(v_1 + c_3)$ and $\lambda = -(v_1 + c_3)(1 + h^2)^{1/2}$, $f = cb_1 - 2c_1\lambda$. Thus, it remains for us to satisfy Eqs. (36) and (38). Having inserted h and λ into (36), we obtain an ordinary differential equation to determine $c(\lambda)$ ($c_1 = \alpha(k^2 - c^2/4)^{1/2}$, $\alpha = \pm 1$):

$$c'(\lambda - \alpha b_1 c/(4k^2 - c^2)^{1/2}) + 2(b + c) = 0.$$

After we find $c = c(\lambda)$ in quadratures we have the general solution of (38)

$$\Phi((x_3 - B\exp(\textstyle\int 2c_1/fd\lambda)\int((c/f)\exp(-\int 2c_1/fd\lambda))d\lambda), B\exp(\int 2c_1/fd\lambda)) = 0$$

with the arbitrary function $\Phi(\xi, \zeta)$.

2. Let us describe the second case, when $v_i = v_i(v_1)$ ($i = 2, 3$) at $S_1 \neq 0$. After we substitute $v_i = v_i(v_1)$ ($i = 2, 3$) into (5), we have

$$\begin{gathered} -S_2 w_1 + S_1 w_2 = 0, \quad -2S_{12}w_1 + S_1(w_2 + v_2' w_1) = 0, \\ (S_1 + S_2)w_1 + S_1 v_3' w_3 = 0, \quad -2S_{13}w_1 + S_1(w_3 + v_3' w_1) = 0, \end{gathered} \tag{46}$$

where for the sake of brevity we introduced the notation $w_i = \partial v_1/\partial x_i$ ($i = 1, 2, 3$). Since system (46) is linear and homogeneous relative to w_i ($i = 1, 2, 3$) and $\sum_i w_i^2 \neq 0$, then

$$\begin{gathered} S_1(v_2')^2 - 2S_{12}v_2' + S_2 = 0, \quad S_1(v_3')^2 - 2S_{13}v_3' - S_1 - S_2 = 0, \\ S_{13}v_2' + S_{12}v_3' - S_1 v_2' v_3' - S_{23} = 0. \end{gathered}$$

Since the solution reduces to plane strain for $v_2' = 0$, it must be assumed that $v_2' \neq 0$. Then we use the last equations to find expressions for S_{12}, S_2, and S_{23}. Inserting them into the von Mises yield condition, we obtain a quadratic equation for S_{13}. It follows from this equation that $|S_1(h + 1)h^{-1/2}/(2k)| \leq 1$ ($h \equiv (v_2')^2 + (v_3')^2$). We introduce an angle θ such that $\sin\theta = S_1(h + 1)h^{-1/2}/(2k)$.

If $S_1 = S_1(v_1)$ (or $\theta = \theta(v_1)$), then $\sigma = \sigma(v_1)$ and the solution is reduced to a simple wave. Thus, we choose S_1 and v_1 as parameters of the double wave. Having inserted $\sigma = \sigma(S_1, v_1)$ into (1) with allowance for the first two independent equations of (46), we write

$$b_{i\alpha}\partial S_1/\partial x_\alpha + b_{i4}w_1 = 0 \quad (i = 1, 2, 3). \tag{47}$$

The form of the coefficients b_{ij} ($i = 1, 2, 3$, $j = 1, 2, 3, 4$) is quite complex and is not presented here. We performed all subsequent calculations on a computer as well, and here we present only the reasoning and the final results.

In order for the double wave to not reduce to an invariant solution in (47), there should be no more than two independent equations. This means that the rank of the matrix $B = (b_{ij})$ must not be greater than two. If we use B_j to represent a square matrix composed of the matrix B without the j-th column, then $\det(B_j) = 0$ ($j = 1, 2, 3, 4$). Equating $\det(B_4)$ to zero, we obtain

$$\partial\sigma/\partial S_1((\partial\sigma/\partial S_1)^2 - (h + 1)^2/(4h\cos^2\theta)) = 0.$$

It follows from this that either $\sigma = \sigma(v_1)$ or $\sigma = \pm S_1(h + 1)h^{-1/2}/(2\cos\theta) + \varphi(v_1)$. Having inserted the expressions for σ into $\det(B_3) = 0$, in both cases we obtain a polynomial in $\tan\theta$ with coefficients dependent on v_1. Since θ and v_1 are assumed to be functionally independent, then these coefficients should vanish. However, there are contradictory equalities among the relations. For example, $h + 1 = 0$. Thus, in the given case, when $v_i = v_i(v_1)$ ($i = 2, 3$), there are no double waves that cannot be reduced to invariant solutions or simple waves.

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THERMOELASTIC STRESSES IN A PLANE WITH A CIRCULAR INCLUSION IN THE PRESENCE OF A THERMAL SPOT OF ELLIPTICAL SHAPE

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The problem of determining the thermostress state of a body during heating by a spot occupying a certain domain reduces to a problem of determining the elastic stresses for given discontinuities in the displacements on the spot boundary [1]. This latter is equivalent to the problem of determining the elastic stresses caused by the presence of an inclusion preliminarily subjected to intrinsic strain and having elastic characteristics as also the surrounding medium and then inserted in the hole occupied by the spot domain [2]. Utilization of the Muskhelishvili method in the plane case permits reducing this problem to a standard boundary value problem of elasticity theory for the whole domain occupied by the body with altered external forces [2]. When the spot is circular in shape, the solution can be found in closed form [3-5]. The solution of the problem of determining the stresses in a half-plane for an elliptical spot shape and constant magnitude of the heating ΔT is also written in closed form [6]. This paper is devoted to obtaining such a solution for a plane with a circular foreign inclusion for an elliptical spot shape and ΔT = const.

Let an elastic plane with a circular foreign inclusion be heated over a certain domain D bounded by the contour L from an initial temperature T_0 for which there is no stress state to a temperature T_1. It is assumed that the contour L does not intersect the circle L_0 bounding the foreign inclusion and can be a system of nonintersecting closed contours L_j ($j = 1, 2, \ldots, n$). Without limiting the generality, we will consider the contour L to consist of two contours L_1 and L_2 bounding domains D_1^+ and D_2^+ lying entirely within and outside the circle L_0, respectively. The domain lying between the contours L_0 and L_1 is denoted by D_1^- and the domain between L_0 and L_2 by D_2^-. It is known [2] that the stress state that occurs is equivalent to that which occurs in inclusions occupying the domain D_j^+ first subjected to intrinsic strain and from the same material as its external medium, and then installed in holes with the contours L_j ($j = 1, 2, \ldots, n$).

Let us assume the center of the circular foreign inclusion of radius R_0 to be at the origin of the x, y plane, and μ_j, ν_j, α_j to be the shear modulus, Poisson ratio, and coefficient of thermal expansion of the materials of the foreign inclusion ($j = 1$) and its external medium ($j = 2$). We use the Muskhelishvili method to find the stress state. Considering that an ideal contact holds on the common boundary of the inclusion with the medium, the conditions of equality of the normal and tangential stresses as well as the presence of a displacement jump on the interfacial lines of the media caused by the intrinsic strains are written in the form

$$\varphi_0^-(t) + t\overline{\varphi_0^{-\prime}(t)} + \overline{\psi_0^-(t)} = \varphi^-(t) + t\overline{\varphi^{-\prime}(t)} + \overline{\psi^-(t)} + C_1, \quad (1)$$

$$\left(\varkappa_1\varphi_0^-(t) - t\overline{\varphi_0^{-\prime}(t)} - \overline{\psi_0^-(t)}\right)/\mu_1 = \left(\varkappa_2\varphi^-(t) - t\overline{\varphi^{-\prime}(t)} - \overline{\psi^-(t)}\right)/\mu_2 \quad (t \in L_0);$$

$$\varphi_0^+(t) + t\overline{\varphi_0^{+\prime}(t)} + \overline{\psi_0^+(t)} = \varphi_0^-(t) + t\overline{\varphi_0^{-\prime}(t)} + \overline{\psi_0^-(t)} + C_2, \quad (2)$$

$$\varkappa_1\varphi_0^+(t) - t\overline{\varphi_0^{+\prime}(t)} - \overline{\psi_0^+(t)} = \varkappa_2\varphi_0^-(t) - t\overline{\varphi_0^{-\prime}(t)} - \overline{\psi_0^-(t)} + 2\mu_1 g_1(t) \quad (t \in L_1);$$

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